3.10 Fundamental Identities

Identities – true for all values of the variable for which both sides of the equation are defined.

The set of all such values is called the **domain of validity** of the identity.

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Quotient Identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta \qquad \csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$$

Even-odd properties:

sin $(-\theta)$ = - sin θ cos $(-\theta)$ = cos θ tan $(-\theta)$ = - tan θ

csc $(-\theta)$ = - csc θ sec $(-\theta)$ = sec θ cot $(-\theta)$ = - cot θ

cos and sec are even functions, the other four are odd.

Using Identities:

Ex. 1

Find $\sin\theta$ and $\cos\theta$ if $\tan\theta$= 5 and $\cos\theta > 0$.

Solution: Using identities we know that $\tan^2\theta + 1 = \sec^2\theta$

So, $5^2 + 1 = \sec^2\theta = 26$ which means $\sec\theta = \pm\sqrt{26}$.

We also know that $\cos\theta = \dfrac{1}{\sec\theta}$, and since $\cos\theta > 0$, then

$\cos\theta = \dfrac{1}{\sqrt{26}}$.

To find $\sin\theta$, we can use $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

so, $\sin\theta = \tan\theta\cos\theta$ which gives us $\sin\theta = 5 \cdot \dfrac{1}{\sqrt{26}} = \dfrac{5}{\sqrt{26}}$

Simplifying by Factoring and Using Identities

Ex. Simplify the expression $\sin^3 x + \sin x \cos^2 x$.

$$\sin^3 x + \sin x \cos^2 x = \sin x(\sin^2 x + \cos^2 x)$$

$$= \sin x(1) = \sin x$$

Simplifying by Expanding and Using Identities

Simplify the expression $[(\sec x+1)(\sec x-1)]/\sin^2 x$.

Solution:

$$\frac{(\sec x+1)(\sec x-1)}{\sin^2 x}=\frac{\sec^2 x-1}{\sin^2 x}=\frac{\tan^2 x}{\sin^2 x}=\frac{\sin^2 x}{\cos^2 x}\cdot\frac{1}{\sin^2 x}=\frac{1}{\cos^2 x}=\sec^2 x$$

Simplifying by Combining Fractions and Using Identities

Simplify the expression $\frac{\cos x}{1-\sin x}-\frac{\sin x}{\cos x}$.

Solution:

$$\frac{\cos x}{1-\sin x}\cdot\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}\cdot\frac{1-\sin x}{1-\sin x}=\frac{(\cos x)(\cos x)-(\sin x)(1-\sin x)}{(1-\sin x)(\cos x)}$$

$$=\frac{\cos^2 x-\sin x+\sin^2 x}{(1-\sin x)(\cos x)}=\frac{1-\sin x}{(1-\sin x)(\cos x)}=\frac{1}{(\cos x)}=\sec x$$

Solving a Trigonometric Equation

Find all values of x in the interval $[0,2\pi)$ that solve $\frac{\cos^3 x}{\sin x}=\cot x$.

Solution:

$$\frac{\cos^3 x}{\sin x}=\cot x$$

$$\frac{\cos^3 x}{\sin x} = \frac{\cos x}{\sin x}$$

$$\frac{\cos^3 x}{\sin x} - \frac{\cos x}{\sin x} = 0$$

$$\frac{\cos x(\cos^2 x - 1)}{\sin x} = 0 \qquad \cos x(-\sin^2 x) = 0$$

$$\cos x = 0 \qquad \sin x = 0$$

We can't use the solutions for $\sin x = 0$, because it would make the original equation undefined. So, we will just look at the solutions for $\cos x = 0$, which are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

Solving a Trigonometric Equation by Factoring

Find all solutions to the trigonometric equation $2\sin^2 x + \sin x = 1$.

Using substitution we can let $y = \sin x$ and rewrite the equation as

$2y^2 + y = 1$

$2y^2 + y - 1 = 0$ factoring we get $(2y-1)(y+1) = 0$ so,

$2y - 1 = 0$ or $y + 1 = 0$ so $y = \frac{1}{2}$ or $y = -1$

Substituting back to the original equation

$\sin x = \frac{1}{2}$ or $\sin x = -1$ which gives us the solutions $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ on the interval $[0, 2\pi)$.